Square function and heat flow estimates on domains

Oana Ivanovici, Fabrice Planchon

SQUARE FUNCTION AND HEAT FLOW ESTIMATES ON DOMAINS

O.IVANOVICI AND F.PLANCHON

Abstract. The first purpose of this note is to provide a proof of the usual square function estimate on $L^p(\Omega)$. It turns out to follow directly from a generic Mikhlin multiplier theorem obtained by Alexopoulos, which mostly relies on Gaussian bounds on the heat kernel. We also provide a simple proof of a weaker version of the square function estimate, which is enough in most instances involving dispersive PDEs. Moreover, we obtain, by a relatively simple integration by parts, several useful $L^p(\Omega; H)$ bounds for the derivatives of the heat flow with values in a given Hilbert space H.

1. Introduction

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$. Let Δ_D denote the Laplace operator on Ω with Dirichlet boundary conditions, acting on $L^2(\Omega)$, with domain $H^2(\Omega) \cap H^1_0(\Omega)$.

The first result reads as follows:

Theorem 1.1. *Let $f \in C^\infty(\Omega)$ and $\Psi \in C^\infty_0(\mathbb{R}^*)$ such that*

$$\sum_{j\in\mathbb{Z}} \Psi(2^{-2j}\lambda) = 1, \quad \lambda \in \mathbb{R}. \tag{1.1}$$

Then for all $p \in (1,\infty)$ we have

$$\|f\|_{L^p(\Omega)} \approx C_p \Big\| \Big(\sum_{j\in\mathbb{Z}} |\Psi(-2^{-2j}\Delta_D) f|^2 \Big)^{1/2} \Big\|_{L^p(\Omega)}, \tag{1.2}$$

where the operator $\Psi(-2^{-2j}\Delta_D)$ is defined by (3.20) *below.*

Readers who are familiar with functional spaces'theory will have recognized the equivalence $\dot F^{0,2}_p \approx L^p$, where the Triebel-Lizorkin space is defined using the right hand-side of (1.2) as a norm. In other words, $L^p(\Omega)$ and the Triebel-Lizorkin space $\dot F^{0,2}_p(\Omega)$ coincide. Such an equivalence (and much more !) is proven in [25, 26, 27], though one has to reconstruct it from several different sections (functional spaces are defined differently, only the inhomogeneous ones are treated, among other things). As such, the casual user with mostly a PDE background might find it difficult to reconstruct the argument for his own sake without digesting the whole theory. It turns out that the proof of (1.2) follows directly from the classical argument (in $\mathbb{R}^n$) involving Rademacher functions, provided that an appropriate Mikhlin-Hörmander multiplier theorem is available. We will provide details below.

A weaker version of Theorem 1.1 is often used in the context of dispersive PDEs:

2000 *Mathematics Subject Classification.* Primary 35J25,58G11.

Theorem 1.2. *Let $f \in C^\infty(\Omega)$, then for all $p \in [2,\infty)$ we have*

$$\|f\|_{L^p(\Omega)} \leq C_p \Big(\sum_{j\in\mathbb{Z}} \|\Psi(-2^{-2j}\Delta_D) f\|^2_{L^p(\Omega)} \Big)^{1/2}. \tag{1.3}$$

The second part of the present note aims at giving a self-contained proof of (1.3) , with "acceptable" black boxes, namely complex interpolation and spectral calculus. In fact, if one accepts to replace the spectral localization by the heat flow, the proof can be made entirely self-contained, relying only on integration by parts. Our strategy to prove Theorem 1.2 is indeed to reduce matters to an estimate involving the heat flow, by proving almost orthogonality between spectral projectors and heat flow localization; this only requires basic parabolic estimates in $L^p(\Omega)$, together with a little help from spectral calculus.

Remark 1.3. For compact manifolds without boundaries, one may find a direct proof of (1.3) (with Δ_D replaced by the Laplace-Beltrami operator) in [7], which proceeds by reduction to the $\mathbb{R}^n$ case using standard pseudo-differential calculus. Our elementary approach provides an alternative direct proof. However, the true square function bound (1.2) holds on such manifolds, as one has a Mikhlin-Hörmander theorem from [21].

Remark 1.4. One can also adapt all proofs to the case of Neumann boundary conditions, provided special care is taken of the zero frequency (note that on an exterior domain, a decay condition at infinity solves the issue). The Gaussian bound which is required later holds in the Neumann case, see [10, 9].

Remark 1.5. As mentioned before, Theorem 1.2 is useful, among other things, when dealing with L^p estimates for wave or dispersive evolution equations. For such equations, one naturally considers initial data in Sobolev spaces, and spectral localization conveniently reduces matters to data in L^2, and helps with finite speed of propagation arguments. One however wants to sum eventually over all frequencies in l^2, if possible without loss. Recent examples on domains may be found in [16] or [20], as well as in [17].

We now state estimates involving directly the heat flow, which will be proved by direct arguments. It should be noted that for nonlinear applications, it is quite convenient to have bounds on derivatives of spectral multipliers, and such bounds do not follow immediately from the multiplier theorem from [2]. We consider the linear heat equation on Ω with Dirichlet boundary conditions and initial data f

$$\partial_t u - \Delta_D u = 0, \text{ on } \Omega \times \mathbb{R}_+; \quad u|_{t=0} = f \in C^\infty(\Omega); \quad u|_{\partial\Omega} = 0. \tag{1.4}$$

We denote the solution $u(t,x) = S(t)f(x)$, where we set $S(t) = e^{t\Delta_D}$. For the sake of simplicity Δ_D has constant coefficients, but the same method applies in the case when the coefficients belong to a bounded set of C^∞ and the principal part is uniformly elliptic (one may lower the regularity requirements on both the coefficients and the boundary, and a nice feature of the proofs which follow is that counting derivatives is relatively straightforward).

Let us define two operators which are suitable heat flow versions of $\Psi(-2^{-2j}\Delta_D)$:

$$Q_t = \sqrt{t}\nabla S(t) \text{ and } \mathbf{Q}_t \overset{\text{def}}{=} t\partial_t S(t). \tag{1.5}$$

Theorem 1.6. *Let $1 < p < +\infty$, then we have*

$$\|f\|_{L^p(\Omega)} \approx c_{,p} \| \Big(\int_0^\infty |Q_t f|^2 \frac{dt}{t} \Big)^{1/2} \|_{L^p(\Omega)}, \tag{1.6}$$

which implies, for $p \in [2, +\infty)$,

$$\|f\|_{L^p(\Omega)} \leq C_p \Big(\int_0^\infty \|Q_t f\|^2_{L^p(\Omega)} \frac{dt}{t} \Big)^{1/2}, \tag{1.7}$$

and Q_t may be replaced by $\mathbf{Q}_t$ in both statements.

Notice that there is no difficulty to define $Q_t f$ or $\mathbf{Q}_t f$ as distributional derivatives for $f \in L^p(\Omega)$, while simply defining $\Psi(-2^{-2j}\Delta_D)$ on $L^p(\Omega)$ is already a non trivial task. The purpose of the next Proposition is to prove that both operators are in fact bounded on $L^p(\Omega)$.

Proposition 1.7. Let $1 < p < +\infty$. The operators Q_t, $\mathbf{Q}_t$ are bounded on $L^p(\Omega)$, uniformly in $t \geq 0$. Moreover $\mathbf{Q}_t$ is bounded on $L^1(\Omega)$ and $L^\infty(\Omega)$.

For practical applications, one may need a vector valued version of Proposition 1.8. Let us consider now $u = (u_l)_{l \in \{1,..,N\}}$ for $N \geq 2$, where each u_l solves (1.4) with Dirichlet condition and initial data f_l. Let H be the Hilbert space with norm $\|u\|_H^2 = \sum_l |u_l|^2$, and $L^p(\Omega; H)$ the Hilbert valued Lebesgue space. Then we have

Proposition 1.8. Let $1 < p < +\infty$. The operators Q_t, $\mathbf{Q}_t$ are bounded on $L^p(\Omega; H)$, uniformly in $t \geq 0$ and N. Moreover $\mathbf{Q}_t$ is bounded on $L^1(\Omega; H)$ and $L^\infty(\Omega; H)$.

Remark 1.9. One may therefore extend the finite dimensional case to any separable Hilbert space. The typical setting would be to consider the solution u to the heat equation with initial data $f(x, \theta) \in L^2_\theta = H$. Notice that the Hilbert valued bound does not follow from the previous scalar bound; however the argument is essentially the same, replacing $|\cdot|$ norms by Hilbert norms.

Remark 1.10. A straightforward consequence of Propositions 1.8 and 1.7 is that the Riesz transforms $\partial_j(-\Delta_D)^{-\frac{1}{2}}$ are continuous on Besov spaces defined by the RHS of (1.7); these spaces are equivalent to the ones defined by the RHS of (1.3), see Remark 3.5 later on.

Alternatively, one can derive all the (scalar, at least) results on the heat flow from adapting to the domain case the theory which ultimately led to the proof of the Kato conjecture ([6, 5]). Such a possible development is pointed out by P. Auscher in [4] (chap. 7, p. 66) and was originally our starting point; eventually we were led to the elementary approach we present here, but we provide a sketch of an alternate proof in the next remark, which was kindly outlined to us by Pascal Auscher.

Remark 1.11. The main drawback from (1.7) is the presence of $\nabla S(t)$ on the right hand-side: one is leaving the functional calculus of Δ_D, and in fact for domains with Lipschitz boundaries the operator $\nabla S(t)$ may not even be bounded. As such, a suitable alternative is to replace $\nabla S(t)$ by $\sqrt{\partial_t} S(t)$. Then the square function estimate may be obtained following [4] as follows:

- prove that the associated square function in time is bounded by the L^p norm, for all $1 < p \leq 2$, essentially following step 3 in chapter 6, page 55 in [4]. This requires very little on the semi-group, and Gaussian bounds on $S(t)$ and $\partial_t S(t)$ ([11]) are more than enough to apply the weak $(1,1)$ criterion from [4] (Theorem 1.1, chapter 1). Moreover, the argument can be extended to domains with Lipschitz boundaries, assuming the Laplacian is defined through the associated Dirichlet form;

- by duality, we get the square function bound for $p > 2$ (step 5, page 56 in [4]);
- from now on one proceeds as in the remaining part of our paper to obtain the bound with spectral localization, and almost orthogonality (3.7) is even easier because we stay in the functional calculus. One has, however, to be careful if one is willing to extend this last step to Lipschitz boundaries, as this would most likely require additional estimates on the resolvent to deal with the Δ_j.

2. From a Mikhlin multiplier theorem to the square function

The following "Fourier multiplier" theorem is obtained in [2] under very weak hypothesis on the underlying manifold (see also [3] for a specific application to Markov chains, and [23] for a version closer to the sharp Hörmander's multiplier theorem, under suitable additional hypothesis, all of which are verified on domains). For $m \in L^\infty(\mathbb{R}^+)$, one usually defines the operator $m(-\Delta_D)$ on $L^2(\Omega)$ through the spectral measure dE_λ:

$$m(-\Delta_D) = \int_0^{+\infty} dE_\lambda, \tag{2.1}$$

and $m(-\Delta_D)$ is bounded on L^2.

Remark 2.1. One may alternatively use the Dynkin-Helffer-Sjöstrand formula as in the Appendix, and both definitions are known to coincide on $L^2(\Omega)$. However, the Dynkin-Helffer-Sjöstrand formula seems to be restricted to defining $m(-\Delta_D)$ for functions m which exhibit slightly more decay than required in the next theorem, at least if one proceeds as exposed in the Appendix.

Theorem 2.2 ([2])**.** *Let $m \in C^N(\mathbb{R}^+)$, $N \in \mathbb{N}$ and $N \geq n/2+1$, such that*

$$\sup_{\xi, k \leq N} |\xi \partial_\xi^k m(\xi)| < +\infty. \tag{2.2}$$

Then the operator defined by (2.1) *extends to a continuous operator on $L^p(\Omega)$, and sends $L^1(\Omega)$ to weak $L^1(\Omega)$.*

In order to use the argument of [2], we need the Gaussian upper bound on the heat kernel, which is provided in our case by [10]. Once we have Theorem 2.2, all we need to do to prove Theorem 1.1 is to follow Stein's classical proof from [22][1], and we recall it briefly for the convenience of the reader. Let us introduce the Rademacher functions, which are defined as follows:

- the function $r_0(t)$ is defined by $r_0(t) = 1$ on $[0, 1/2]$ and $r_0(t) = -1$ on $(1/2, 1)$, and then extended to $\mathbb{R}$ by periodicity;
- for $m \in \mathbb{N} \setminus \{0\}$, $r_m(t) = r_0(2^m t)$.

[1]we thank Hart Smith for bringing this to our attention

Their importance is outlined by the following inequalities (see the Appendix in [22]),

$$c_p\|\sum_m a_m r_m(t)\|_{L^p_t} \leq (\sum_m |a_m|^2)^{\frac{1}{2}} \leq C_p\|\sum_m a_m r_m(t)\|_{L^p_t}. \tag{2.3}$$

Now, define

$$m^{\pm}(t,\xi) = \sum_{j=0}^{+\infty} r_j(t)\Psi_{\pm j}(\xi),$$

where Ψ_j was defined in the introduction. A straightforward computation proves that the bound (2.2) holds for $m^{\pm}(t,\xi)$. Therefore,

$$\|m^{\pm}(t,-\Delta_D)f\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)};$$

integrating in time over $[0,1]$, exchanging space and time norms, and using (2.3),

$$\|m^{\pm}(t,-\Delta_D)f\|_{L^p(\Omega)L^2(0,1)} \approx \|\big(\sum_{j=0}^{\pm\infty} |\Psi(-2^{-2j}\Delta_D)f|^2\big)^{\frac{1}{2}}\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}.$$

This proves one side of the equivalence in (1.2): the other side follows from duality, once we see the above estimate as an estimate from $L^p(\Omega)$ to $L^p(\Omega;l^2)$, which maps f to $(\Psi(-2^{-2j}\Delta_D)f)_{j\in\mathbb{Z}}$.

3. Heat flow estimates

In order to prove Proposition 1.6 we need the following lemma.

Lemma 3.1. *For all* $1 \leq p \leq +\infty$*, we have*

$$\|S(t)f\|_{L^p(\Omega)} \to_{t\to\infty} 0, \tag{3.1}$$

$$\sup_{t\geq 0} \|S(t)f\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}\,. \tag{3.2}$$

Moreover,

$$\|\sup_{t\geq 0} |S(t)f|\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}\,. \tag{3.3}$$

Proof: The estimate (3.2) clearly follows from (3.3), which in turn is a direct consequence of the Gaussian nature of the Dirichlet heat kernel, see [10]. The same Gaussian estimate implies (3.1). However we do not need such a strong fact to prove (3.2), which will follow from the next computation as well (see (3.4)) when $1 < p < +\infty$. Estimate (3.1) can also be obtained through elementary arguments. We defer such a proof to the end of the section.

3.1. Proof of Theorem 1.6.

If $p = 2$ the proof is nothing more than the energy inequality, combined with (3.1). In fact, for $p = 2$, we have equality in (1.6) with $C_2 = 2$. We now take $p = 2m$ where $m \geq 2$. Multiplying equation (1.4) by $\bar{u}|u|^{p-1}$ and taking the integral over Ω and $[0,T]$, $T > 0$ yields, taking advantage of the Dirichlet boundary condition,

$$\frac{1}{p}\int_0^T \partial_t\|u\|^p_{L^p(\Omega)}dt + \int_0^T\int_\Omega |\nabla u|^2|u|^{p-2}dxdt + \\ + \frac{(p-2)}{2}\int_0^T\int_\Omega (\nabla(|u|^2))^2|u|^{p-4}dxdt = 0, \tag{3.4}$$

from which we can estimate either $\|u\|^p_{L^p(\Omega)}(T) \leq \|f\|^p_{L^p(\Omega)}$ (which is (3.2)) or

$$\|f\|^p_{L^p(\Omega)} \leq \|u\|^p_{L^p(\Omega)}(T) + p(p-1)\int_0^T \int_\Omega |\nabla u|^2 |u|^{p-2} dxdt.$$

Letting T go to infinity and using (3.1) from Lemma 3.1 and Hölder inequality we find

$$\|f\|^p_{L^p(\Omega)} \leq p(p-1)\Big(\int_\Omega \Big(\int_0^\infty |\nabla u|^2\, dt\Big)^{\frac{p}{2}} dx\Big)^{\frac{2}{p}} \Big(\int_\Omega (\sup_t |u|^{p-2})^{\frac{p}{p-2}} dx\Big)^{\frac{p-2}{p}}.$$

The proof follows using again Lemma 3.1, as

$$\|f\|^p_{L^p(\Omega)} \leq C_p \|\Big(\int_0^\infty |\nabla u|^2 dt\Big)^{\frac{1}{2}}\|_{L^p(\Omega)} \left(\|\sup_{t\geq 0}|u|\|_{L^p(\Omega)}\right)^{p-2}.$$

Note that we may prove the weaker part, (1.7), without assuming the maximal in time bound, by reversing the order of integration in our argument. This would keep the argument for heat square functions essentially self-contained, without any need for Gaussian bounds on the heat kernel.

Remark 3.2. We do not claim novelty here: our argument follows closely (a dual version of) the proof of a classical square function bound for the Poisson kernel in the whole space, see [22].

We have proved one side of the equivalence in (1.6) involving the Q_t square function, in the range $2 \leq p < +\infty$; we now prove the other side, by duality. Let $\phi \in L^q(\Omega)$, with $1/q = 1 - 2/p$, and consider

$$I = \int_\Omega \big(\int_0^{+\infty} |\nabla u|^2\, dt\big)\phi(x)\, dx.$$

Without any loss of generality, we may assume $\phi \geq 0$. On the other hand, let $v = |\nabla u|^2$, then

$$\partial_t v - \Delta v = -2|\nabla^2 u|^2,$$

and one checks easily that $\partial_n v = 0$ on $\partial\Omega$. Let $S_n(t)$ be the solution to the heat equation on Ω with Neumann boundary condition, by comparing v and $S_n(t/2)v(t/2)$ (formally, take the difference, multiply by the positive part and integrate by parts) we have

$$0 \leq v \leq S_n(t/2)v(t/2) = S_n(t/2)|\nabla u(t/2)|^2,$$

and therefore

$$I \leq 2\int_\Omega \int_0^{+\infty} |\nabla u|^2\, S_n(t)\phi\, dxdt. \tag{3.5}$$

Now, we also have

$$\partial_t u^2 - \Delta_D u^2 = -2|\nabla u|^2,$$

and therefore

$$I \leq -\int_\Omega \int_0^{+\infty} (\partial_t - \Delta)(u^2) S_n(t)\phi\, dxdt.$$

From $(\partial_t - \Delta)(u^2 S_n(t)\phi) = -2\nabla(u^2)\cdot\nabla S_n(t)\phi$, we get

$$I \leq \int_\Omega 4\sup_t |u| \big(\int_0^{+\infty} |Q_t u|^2 \frac{dt}{t}\big)^{\frac{1}{2}} \big(\int_0^{+\infty} |\nabla S_n(t)\phi|^2\, dt\big)^{\frac{1}{2}}.$$

The bound we already proved with Q_t can easily be reproduced with $S(t)$ replaced by $S_n(t)$, and therefore, provided $q \leq 2$, we may use the dual bound on the square function of $\phi \in L^q(\Omega)$ and conclude by Hölder, using (3.3) on the first factor. The condition on q translates into $p \geq 4$, and the remaining $2 < p < 4$ are handled by interpolation.

Remark 3.3. Actually, we may directly bound $S_n(t)$ by a Gaussian in (3.5), extend ϕ by 0 outside Ω, and use the heat square function bounds in $\mathbb{R}^n$. This provides a direct argument, irrespective of the value of p.

It remains to prove the equivalence between the Q_t square function and the $\mathbf{Q}_t$ square function. For this, we repeat the duality argument but we replace $|\nabla u|^2$ by $t|\partial_t u|^2$. Notice that $\partial_t u$ is also a solution to the heat equation with Dirichlet boundary condition, and if $w = |\partial_t u|^2$,

$$(\partial_t - \Delta)w = -2|\nabla \partial_t u|^2.$$

Therefore, comparing w and $S(t/2)w(t/2)$,

$$0 \leq |\partial_t u|^2 \leq S(t/2)|\partial_t u(t/2)|^2,$$

and

$$J = \int_\Omega \int_0^{+\infty} |\partial_t u|^2 t\phi \, dxdt \leq 2 \int_\Omega \int_0^{+\infty} |\partial_t u|^2 t S(t)\phi \, dxdt.$$

Now,

$$\begin{aligned} J &\leq 2 \int_\Omega \int_0^{+\infty} t\partial_t u \Delta u S(t)\phi \, dxdt \\ &\leq - \int_\Omega \int_0^{+\infty} t\partial_t |\nabla u|^2 S(t)\phi \, dxdt - 2 \int_\Omega \int_0^{+\infty} t\partial_t u \nabla u \nabla S(t)\phi \, dxdt \\ &\leq \int_\Omega \int_0^{+\infty} |\nabla u|^2 (1 + t\partial_t) S(t)\phi \, dxdt + 2 \int_\Omega \int_0^{+\infty} |\mathbf{Q}_t u Q_t u Q_t \phi| \, \frac{dt}{t} \, dx \end{aligned}$$

from which we can easily conclude by Hölder (using Lemma 1.8 to bound $t\partial_t S(t)\phi$). Duality takes care of the reverse bound, and this concludes the proof of Theorem 1.6, except for the equivalence between the Q_t and $\mathbf{Q}_t$ Besov norms in (1.7); we defer this to the end of the next subsection.

Notice that, at this point, we proved Theorem 1.2, but with the Ψ operator replaced by the gradient heat kernel and the discrete parameter 2^{-2j} by the continuous parameter t. The rest of this section is devoted to proving the equivalence between the Besov norms which are defined by the heat kernel or the spectral localization.

Lemma 3.4. *Let $1 \leq p \leq +\infty$. We have the following equivalence between dyadic and continuous versions of the Besov norm:*

$$\frac{3}{4} \sum_{k \in \mathbb{Z}} \|Q_{2^{-2k}} f\|^2_{L^p(\Omega)} \leq \int_0^\infty \|Q_t f\|^2_{L^p(\Omega)} \frac{dt}{t} \leq 3 \sum_{k \in \mathbb{Z}} \|Q_{2^{-2k}} f\|^2_{L^p(\Omega)}.$$

This follows at once from factoring the semi-group: for $2^{-2j} \leq t \leq 2^{-2(j-1)}$, write $S(t) = S(t - 2^{-2j})S(2^{-2j})$ and use (3.2). We now turn to the direct proof of Theorem 1.2 from the heat flow version. Let $\Psi \in C_0^\infty(\mathbb{R}^*)$ satisfying (1.1) and denote $\Delta_j f \overset{\text{def}}{=} \Psi(2^{-2j}\Delta_D)f$, where $\Psi(2^{-2j}\Delta_D)f$ is given by the Dynkin-Helffer-Sjöstrand formula (see the Appendix, (3.20)). From Proposition 1.6 and Lemma 3.4 we have

$$\|f\|_{L^p(\Omega)} \leq 3C_p\Big(\sum_{k\in\mathbb{Z}} \|Q_{2^{-2k}}f\|^2_{L^p(\Omega)}\Big)^{1/2} \tag{3.6}$$

and we will show that (3.6) implies (1.2): it suffices to prove the following almost orthogonality property between localization operators Δ_j and $Q_{2^{-2k}}$:

$$\forall k, j \in \mathbb{Z}, \qquad \|Q_{2^{-2k}}\Delta_j f\|_{L^p(\Omega)} \lesssim 2^{-|j-k|}\|\Delta_j f\|_{L^p(\Omega)}. \tag{3.7}$$

Then, from $(2^{-|j-k|})_k \in l^1$ and $(\|\Delta_j f\|_{L^p(\Omega)})_j \in l^2$ we estimate

$$\sum_{k\in\mathbb{Z}} \|Q_{2^{-2k}}f\|^2_{L^p(\Omega)} = \sum_{k\in\mathbb{Z}} \|\sum_{j\in\mathbb{Z}} Q_{2^{-2k}}\Delta_j f\|^2_{L^p(\Omega)} \tag{3.8}$$

as an $l^1 * l^2$ convolution and conclude using Lemma 1.8. It remains to show (3.7):

- for $k < j$ we write

$$Q_{2^{-2k}}\Delta_j f = 2^{3/2}2^{-2(j-k)}\Big(2^{-(2k+1)/2}\nabla S(2^{-(2k+1)})\Big) \\ \Big(2^{-(2k+1)}\Delta_D S(2^{-(2k+1)})\Big)\breve{\Psi}(-2^{-2j}\Delta_D)\Psi(-2^{-2j}\Delta_D)f,$$

  where we set $\breve{\Psi}(\lambda) \overset{\text{def}}{=} \frac{1}{\lambda}\tilde{\Psi}(\lambda)$, and $\tilde{\Psi} \in C_0^\infty$, $\tilde{\Psi} = 1$ on suppΨ. By Lemma 1.8, the operators $Q_{2^{-(2k+1)}} = 2^{-(2k+1)/2}\nabla S(2^{-(2k+1)})$ and $\mathbf{Q}_{2^{-(2k+1)}} = 2^{-(2k+1)}\Delta_D S(2^{-(2k+1)})$ are bounded on $L^p(\Omega)$ and we obtain (3.7) using Corollary 3.12 for $\breve{\Psi}$.
- for $k \geq j$ we set $\Psi_1(\xi) = \tilde{\Psi}(\xi)\exp(\xi)$, $\Psi_2(\xi) = \Psi(\xi)$, and we use again Lemma 3.15 to write (slightly abusing the notation as $2^{-2k} - 2^{-2j} < 0$)

$$S(2^{-2k} - 2^{-2j})\Delta_j f = S(2^{-2k})\Psi_1(-2^{-2j}\Delta_D)\Psi_2(-2^{-2j}\Delta_D)f. \tag{3.9}$$

  Then

$$Q_{2^{-2k}}\Delta_j f = 2^{-(k-j)}\Big(2^{-j}\nabla S(2^{-2j})\Big)\Big(S(2^{-2k} - 2^{-2j})\Delta_j f\Big),$$

  and using again Lemma 1.8 we see that the operator $2^{-j}\nabla S(2^{-2j})$ is bounded while the remaining operator (3.9) is bounded by Corollary 3.12. This ends the proof.

Remark 3.5. One may prove a similar bound with $Q_{2^{-2k}}$ and Δ_j reversed, either directly or by duality. Hence Besov norms based on Δ_j or $Q_{2^{-2k}}$ are equivalent.

3.2. **Proof of Proposition 1.8.** For $\mathbf{Q}_t$, boundedness on all L^p spaces, including $p = 1, +\infty$, follows once again from a Gaussian upper bound on $\partial_t S(t)$ (see [11] or [13]). However the subsequent Gaussian bound on the gradient $\nabla_x S(t)$ in [11] is a direct consequence of the Li-Yau inequality, which holds only inside convex domains. We were unable to find a reference which would provide the desired bound for Q_t in the context of the exterior domain. Therefore we provide an elementary detailed proof for Q_t. Furthermore, we only deal with $1 < p < 2$ or powers of two, $p = 2^m$, $m \in \mathbb{N}^*$: complex interpolation takes care of remaining values of p, though one could adapt the following argument to generic values $p > 2$, at the expense of lengthier computations.

Set $v(x,t) = (v_1, .., v_n)(x,t) := Q_t f = t^{1/2}\nabla u(x,t)$ and assume without loss of generality that v_j are real: we multiply the equation satisfied by v by $v|v|^{p-2}$, where $|v|^2 = \sum_{j=1}^n v_j^2$, and integrate over Ω,

$$\partial_t \left(\frac{1}{p}\|v\|^p_{L^p(\Omega)}\right) - \sum_{j=1}^n \int_{\partial\Omega} ((\overrightarrow{\nu}\cdot\nabla)v_j)\cdot v_j|v|^{p-2}d\sigma + \\ + \int_\Omega |\nabla v|^2|v|^{p-2}\,dx + \frac{(p-2)}{2}\int_\Omega \nabla(|v|^2)|v|^{p-4}\,dx = \frac{1}{2t}\|v\|^p_{L^p(\Omega)}, \tag{3.10}$$

where $\overrightarrow{\nu}$ is the outgoing unit normal vector to $\partial\Omega$ and $d\sigma$ is the surface measure on $\partial\Omega$. We claim that the second term in the left hand side vanishes: in fact we write

$$\sum_{j=1}^n \int_{\partial\Omega} (\overrightarrow{\nu}\cdot\nabla v_j)\cdot v_j|v|^{p-2}d\sigma = \\ = \frac{t^{p/2}}{2}\int_{\partial\Omega} \partial_\nu(|\partial_\nu u|^2 + |\nabla_{tang}u|^2)(|\partial_\nu u|^2 + |\nabla_{tang}u|^2)^{(p-2)/2}d\sigma, \tag{3.11}$$

and from $u|_{\partial\Omega} = 0$ the time and tangential derivative $(\partial_t, \nabla_{\mathrm{tang}})u|_{\partial\Omega}$ vanishes; furthermore, using the equation, $\partial_\nu^2 u = 0$ on $\partial\Omega$.

Remark 3.6. Notice that while this term does not vanish with Neumann boundary conditions, it will be a lower order term (like $|\nabla u|^2$ on $\partial\Omega$) which can be controled by the trace theorem.

Now, if $1 < p < 2$, multiply by $\|v\|^{2-p}_{L^p(\Omega)}$ and integrate over $[0,T]$,

$$\|v\|^2_{L^p(\Omega)}(T) \lesssim \int_0^T \|Q_t f\|^2_{L^p(\Omega)}\frac{dt}{t} \lesssim \|f\|_p^2,$$

where the last inequality is the dual of (1.6). Hence we are done with $1 < p < 2$.

Remark 3.7. We ignored the issue of v vanishing in the third term in (3.10). This is easily fixed by replacing $|v|^{p-2}$ by $(\sqrt{\varepsilon + |v|^2})^{p-2}$ and proceeding with the exact same computation. Then let ε go to 0 after dropping the positive term on the left handside of (3.10).

Now let $p = 2^m$ with $m \geq 1$: we proceed directly by integrating (3.10) over $[0,T]$, to get

$$\frac{1}{p}\|v\|^p_{L^p(\Omega)}(T) + \int_0^T\int_\Omega |\nabla v|^2|v|^{p-2}\,dxdt + \\ + \frac{(p-2)}{2}\int_0^T\int_\Omega |\nabla(|v|^2)|^2|v|^{p-4}\,dxdt = \int_0^T \frac{1}{2t}\|v\|^p_{L^p(\Omega)}\,dt. \tag{3.12}$$

On the other hand (recall (3.4)),

$$\frac{1}{p}\|u\|^p_{L^p(\Omega)}(T) + (p-1)\int_0^T\int_\Omega |\nabla u|^2|u|^{p-2}\,dxdt = \frac{1}{p}\|f\|^p_{L^p(\Omega)}. \tag{3.13}$$

If $p = 2$ the estimates are trivial since from (3.12), (3.13),

$$\frac{1}{2}\|v\|^2_{L^2(\Omega)}(T) \leq \int_0^T \frac{1}{2t}\|v\|^2_{L^2(\Omega)}dt = \frac{1}{2}\int_0^T \|\nabla u\|^2_{L^2(\Omega)}\,dt \leq \frac{1}{4}\|f\|^2_{L^2(\Omega)}.$$

Now, let $p \geq 4$; for convenience, denote by J the second integral in the left hand-side of (3.12) (notice that the third integral is bounded from above by J), hence

$$J = \int_0^T \int_\Omega |\nabla^2 u|^2 |\nabla u|^{p-2} t^{\frac{p}{2}}\, dxdt = \int_0^T \int_\Omega (\sum_{i,j} |\partial^2_{i,j} u|^2)(\sum_j |\partial_j u|^2)^{\frac{(p-2)}{2}} t^{\frac{p}{2}}\, dxdt,$$

and set

$$I_k = \int_0^T \int_\Omega |\nabla u|^{2k} |u|^{p-2k} t^{k-1}\, dxdt \text{ where } 2 \leq 2k \leq p. \tag{3.14}$$

For our purposes, it suffices to estimate the right hand-side of (3.12), which rewrites

$$\frac{1}{2}\int_0^T t^{\frac{p}{2}-1} \|\nabla u\|^p_{L^p(\Omega)}\, dt = \frac{1}{2} I_{\frac{p}{2}}. \tag{3.15}$$

Integrate by parts the inner (space) integral in I_k, the boundary term vanishes and collecting terms,

(3.16)

$$\int_\Omega \nabla u \nabla u |\nabla u|^{2(k-1)} |u|^{p-2k}\, dx \;\leq\; \frac{(2k-1)}{(p-2k+1)} \int_\Omega |\nabla^2 u| |\nabla u|^{2k-2} |u|^{p-2k+1} dx.$$

By Cauchy-Schwarz the integral in the right hand side of (3.16) is bounded by

$$\Big(\int_\Omega |\nabla^2 u|^2 |\nabla u|^{p-2} dx\Big)^{1/2} \Big(\int_\Omega |\nabla u|^{4k-4-(p-2)} |u|^{2p+2-4k} dx\Big)^{1/2},$$

therefore for $k \geq \frac{p}{4}+1$ we have

$$I_k \lesssim \frac{(2k-1)}{(p-2k+1)} J^{\frac{1}{2}} I^{\frac{1}{2}}_{2k-\frac{p}{2}-1}.$$

We aim at controlling I_m by $J^{1-\eta} I_1^\eta$, for some $\eta > 0$ which depends on m (notice that when $p = 4$, which is $m = 2$, we are already done, using $k = 2$!). Set $k = \frac{p}{2} - (2^j - 1)$ with $j \leq m-2$,

$$I_{2^{m-1}-(2^j-1)} \leq \frac{(2^m - (2^{j+1}-1))}{(2^{j+1}-1)} J^{\frac{1}{2}} I^{\frac{1}{2}}_{2^{m-1}-(2^{j+1}-1)},$$

and iterating $m-2$ times, we finally control $I_{\frac{p}{2}}$ by $J^{1-\eta} I_1^\eta$, which proves that Q_t is bounded on $L^p(\Omega)$.

We now proceed to obtain boundedness of $\mathbf{Q}_t$ on $L^p(\Omega)$ from the Q_t bound; this is worse than using the Gaussian properties of its kernel, as the constants blow up when $p \to 1, +\infty$. It is, however, quite simple. By duality $Q_t^\star$ is bounded on $L^p(\Omega)$, and

$$\mathbf{Q}_t = t\partial_t S(t) = tS(\frac{t}{2})\Delta S(\frac{t}{2}) = 2\sqrt{\frac{t}{2}} S(\frac{t}{2}) \nabla \cdot \sqrt{\frac{t}{2}} \nabla S(\frac{t}{2}) = 2 Q^\star_{\frac{t}{2}} Q_{\frac{t}{2}},$$

and we are done with Lemma 1.8.

From the previous decomposition, we also obtain

$$\|\mathbf{Q}_t f\|_{L^p(\Omega)} \lesssim \|Q_t f\|_{L^p(\Omega)},$$

which implies that any Besov norm defined with $\mathbf{Q}_t$ is bounded by the corresponding norm for Q_t. The reverse bound is true as well, though slightly more involved. We provide the proof for completeness. Consider $f, h \in C_0^\infty(\Omega)$ and $\langle f, g\rangle = \int_\Omega fg$. Then

$$\begin{aligned}\langle f, g\rangle &= -\int_0^{+\infty} \langle \partial_t S(t) f, h\rangle\, dt = -2\int_0^{+\infty} \langle \partial_t S(t) f, S(t)h\rangle\, dt\\ &= 2\int_{t<s} \langle \partial_t S(t) f, \partial_s S(s) h\rangle\, dtds = 4\int_0^{+\infty} \langle \nabla S(s)\partial_t S(t) f, \nabla S(s) h\rangle\, dtds\\ &\lesssim \int_s \left\| \int_0^s \nabla S(t)\partial_s S(s) f\, dt \right\|_p \|\nabla S(s) h\|_{p'}\, ds \lesssim \int_s \sqrt{s}\|\partial_s S(s) f\|_p \|\nabla S(s) h\|_{p'}\, ds\end{aligned}$$

where we used our bound on $\sqrt{t}\nabla S(t)$ at fixed t. Then

$$\langle f, h\rangle \lesssim \int_s \|\mathbf{Q}_s f\|_p \|Q_s h\|_{p'} \frac{ds}{s}$$

from which we are done by Hölder.

3.3. **Proof of Proposition 1.7.** Let us consider now the vector valued case $u = (u_l)_{l\in\{1,..,N\}}$ for $N \geq 2$, where each u_l solves (1.4) with Dirichlet condition and initial data f_l. For the sake of simplicity we consider only real valued u_l, and write

$$|u|^2 = \sum_{l=1}^N u_l^2,\ \ |\nabla u_l|^2 = \sum_{j=1}^n (\partial_j u_l)^2,\ \ |\nabla u|^2 = \sum_{j=1}^n \sum_{l=1}^N (\partial_j u_l)^2$$

Notice that n is the spatial dimension and is fixed through the argument: hence all constants may depend implicitely on n, while N is the dimension of H. For $p = 1, +\infty$, the boundedness of $\mathbf{Q}_t$ follows from the Gaussian character of the time derivative heat kernel, which is diagonal on H.

We proceed with Q_t. Multiplying the equation satisfied by u_l by $u_l|u|^{p-2}$, integrating over Ω and summing up we immediately get (3.4). We now proceed to obtain bounds for $v(x,t) = (v_l(x,t))_l$, where $v_l(x,t) = t^{1/2}\nabla u_l(x,t)$. Multiplying the equation satisfied by v_l by $v_l|v|^{p-2}$ where $|v|^2 = t|\nabla u|^2$, summing up over l and taking the integral over Ω yields

$$\begin{aligned}(3.17)\quad & \frac{1}{p}\|v\|^p_{L^p(\Omega)}(T) + \sum_{l=1}^N\sum_{j=1}^n \int_0^T\int_\Omega |\nabla(\partial_j u_l)|^2|\nabla u|^{p-2}\, dxdt+\\ &+\frac{(p-2)}{4}\int_0^T\int_\Omega |\nabla|\nabla u|^2|^2|\nabla u|^{p-4}t^{p/2}\, dxdt = \int_0^T \|\nabla u\|^p_{L^p(\Omega)} t^{p/2-1}\, dt = \frac{1}{2} I_{\frac{p}{2}},\end{aligned}$$

where $|\nabla(\partial_j u_l)|^2 = \sum_{i=1}^n (\partial^2_{i,j} u_l)^2$, $|\nabla u|^2 = \sum_{l=1}^N \sum_{j=1}^n (\partial_j u_l)^2$. Notice again that the boundary term vanishes. Denote the last two integrals in the left hand side by J_1, J_2. Like before, we perform integrations by parts in I_k defined in (3.14) to obtain

$$\int_\Omega |\nabla u|^{2k}|u|^{p-2k}\,dx = -\sum_{l=1}^{N}\int_\Omega u_l\Delta u_l|\nabla u|^{2(k-1)}|u|^{p-2k}\,dx- \tag{3.18}$$
$$-(k-1)\sum_{l=1}^{N}\int_\Omega u_l\nabla u_l\nabla(|\nabla u|^2)|\nabla u|^{2(k-2)}|u|^{p-2k}\,dx-$$
$$-(p-2k)\sum_{i=1}^{n}\int_\Omega(\sum_{l=1}^{N}\partial_i u_l u_l)^2|u|^{p-2k-2}\,dx.$$

For $k\geq \frac{p}{4}+1$ we estimate the first term in the right hand side of (3.18) by

$$\int_\Omega(\sum_{l=1}^{N}u_l^2)^{1/2}(\sum_{l=1}^{N}(\Delta u_l)^2)^{1/2}|\nabla u|^{2(k-1)}|u|^{p-2k}\,dx\leq$$
$$(\sum_{l=1}^{N}\sum_{j=1}^{n}\int_\Omega\int_\Omega|\nabla(\partial_j u_l)|^2|\nabla u|^{p-2}\,dx)^{1/2}(\int_\Omega|\nabla u|^{4k-p-2}|u|^{2p-4k+2}\,dx)^{1/2},$$

and the second term in the right hand side of (3.18) by

$$(k-1)(\int_\Omega\sum_{i=1}^{n}(\partial_i(|\nabla u|^2))^2|\nabla u|^{p-4}\,dx)^{1/2}(\int_\Omega|\nabla u|^{4k-p-2}|u|^{2p-4k+2}\,dx)^{1/2},$$

where we used that

$$\sum_{l=1}^{N}u_l\nabla u_l\nabla(|\nabla u|^2)\leq\sum_{i=1}^{n}(\sum_{l=1}^{N}u_l^2)^{1/2}(\sum_{l=1}^{N}(\partial_i u_l)^2)^{1/2}|\partial_i(|\nabla u|^2)|\lesssim|u||\nabla u||\nabla(|\nabla u|^2)|.$$

Since the last term in (3.18) is negative, while the quantity we want to estimate is positive we obtain from the last inequalities

$$\int_0^T\int_\Omega|\nabla u|^{2k}|u|^{p-2k}t^{k-1}\,dxdt\lesssim(J_1^{1/2}+J_2^{1/2})I_{2k-\frac{p}{2}-1}\lesssim(J_1+J_2)^{1/2}I_{2k-\frac{p}{2}-1}. \tag{3.19}$$

From now on we proceed exactly like in the scalar case iterating sufficiently many times to obtain the desired result, since we control $I_{p/2}$ which is the RHS term of (3.18) using (3.19).

3.4. **A simple argument for** (3.1)**.** We now return to the first estimate in Lemma 3.1: while we only deal with $p=2$, there is nothing specific to the L^2 case in what follows. Let χ be a smooth cut-off near the boundary $\partial\Omega$. Then $v=(1-\chi)u$ solves the heat equation in the whole space, with source term $[\chi,\Delta]u$:

$$(1-\chi)u=S_0(t)(1-\chi)u_0+\int_0^t S_0(t-s)[\chi,\Delta]u(s)\,ds,$$

where S_0 is the free heat semi-group. We have, taking advantage of the localization near the boundary,

$$\|[\chi,\Delta]u\|_{L^2_t(L^{\frac{2n}{n+2}})}\lesssim C(\chi,\chi')\|\nabla u\|_{L^2_t(L^2)}<+\infty,$$

by the energy inequality (3.4). The integral equation on $(1-\chi)u$ features S_0 for which we have trivial Gaussian estimates, and both the homogeneous and inhomogeneous terms are $C_t(L^2)$ and go to zero as time goes to $+\infty$. On the other hand, by Poincaré inequality (or Sobolev),

$$\int_0^t \|\chi u\|_2^2\,ds \lesssim \int_0^t \|\nabla u\|_2^2\,ds,$$

which ensures that $\|\chi u\|_2$ goes to zero as well at $t=+\infty$.

Acknowledgments

The authors would like to thank Hart Smith for pointing out the relevance of [2, 3, 23] in this context, Pascal Auscher and Francis Nier for entertaining discussions, not to mention providing material which greatly improved content, and Nikolay Tzvetkov for helful remarks on an early draft. Part of this work was conducted while the second author was visiting the Mittag-Leffler institute, which he is grateful to for its hospitality. Both authors were partially supported by the A.N.R. grant "Equa-disp".

Appendix: functional calculus

We start by recalling the Dynkin-Helffer-Sjöstrand formula ([14, 15]) and refer to the appendix of [18] for a nice presentation of the use of almost-analytic extensions in the context of functional calculus. In what follows we will also rely on Davies'presentation ([12]) from which we will use a couple of useful lemma.

Definition 3.8. (see [18, Lemma A.1]) Let $\Psi\in C_0^\infty(\mathbb{R})$, possibly complex valued. We assume that there exists $\tilde\Psi\in C_0^\infty(\mathbb{C})$ such that $|\bar\partial\tilde\Psi(z)|\leq C|\mathrm{Im} z|$ and $\tilde\Psi|_{\mathbb{R}}=\Psi$. Then we have (as a bounded operator in $L^2(\Omega)$)

$$\Psi(-h^2\Delta_D)=\frac{i}{2\pi}\int_{\mathbb{C}}\bar\partial\tilde\Psi(z)(z+h^2\Delta_D)^{-1}d\bar z\wedge dz. \tag{3.20}$$

The next result ensures the existence of $\tilde\Psi$ in the previous definition (see [18, Lemma A.2] and [24], where it is linked with Hadamard's problem of finding a smooth function with prescribed derivatives at a given point):

Lemma 3.9. *If Ψ belongs to $C_0^\infty(\mathbb{R})$ there exists $\tilde\Psi\in C_0^\infty(\mathbb{C})$ such that $\tilde\Psi|_{\mathbb{R}}=\Psi$ and*

$$|\bar\partial\tilde\Psi(z)|\leq C_{N,\Psi}|\mathrm{Im} z|^N,\quad \forall z\in\mathbb{C},\quad \forall N\in\mathbb{N}. \tag{3.21}$$

Moreover, if Ψ belongs to a bounded subset of $C_0^\infty(\mathbb{R})$ (elements of $\mathcal{B}$ are supported in a given compact subset of $\mathbb{R}$ with uniform bounds), then the mapping $\mathcal{B}\ni\Psi\to\tilde\Psi\in C_0^\infty(\mathbb{C})$ is continuous and $C_{N,\Psi}$ can be chosen uniformly w.r.t $\Psi\in\mathcal{B}$.

Remark 3.10. Estimate (3.21) simply means that $\bar\partial\tilde\Psi(z)$ vanishes at any order on the real axis. Precisely, if $z=x+iy$

$$\partial_y^N\tilde\Psi|_{\mathbb{R}}=(i\partial_x)^N\tilde\Psi|_{\mathbb{R}}=(i\partial_x)^N\Psi|_{\mathbb{R}}.$$

In particular if $\langle x\rangle=(1+x^2)^{1/2}$ then for any given $N\geq 0$, a useful example of an almost analytic extension of $\Psi\in C_0^\infty(\mathbb{R})$ is given by

$$\tilde\Psi(x+iy)=\Big(\sum_{m=0}^N \partial^m\Psi(x)(iy)^m/m!\Big)\tau(\frac{y}{\langle x\rangle}),$$

where τ is a non-negative C^∞ function such that $\tau(s) = 1$ if $|s| \le 1$ and $\tau(s) = 0$ if $|s| \ge 2$. For later purposes, we also set

$$\|\Psi\|_N \stackrel{\text{def}}{=} \sum_{m=0}^{N} \int_{\mathbb{R}} |\partial^m \Psi(x)| \langle x \rangle^{m-1} dx.$$

Our next lemma lets us deal with Lebesgue spaces.

Lemma 3.11. *Let $z \notin \mathbb{R}$ and $|\mathrm{Im} z| \lesssim |\mathrm{Re} z|$, then Δ_D satisfies*

$$\|(z - \Delta_D)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)} \le \frac{c}{|\mathrm{Im} z|}\left(\frac{|z|}{|\mathrm{Im} z|}\right)^\alpha, \quad \forall z \notin \mathbb{R} \tag{3.22}$$

for $1 \le p \le +\infty$, with a constant $c = c(p) > 0$ and $\alpha = \alpha(n,p) > n|\frac{1}{2} - \frac{1}{p}|$.

Remark that, for all $h \in (0,1]$, the operator $h^2\Delta_D$ satisfies (3.22) *with the same constants c and α (this is nothing but scale invariance).*

For $p = 2$ the proof of Lemma 3.11 is trivial by multiplying the resolvent equation $-\Delta_D u + zu = f$ by $\bar{u}$ and we get $\alpha = 0$; however for $p \ne 2$ it requires a non trivial argument which we postpone to the end of this Appendix.

Corollary 3.12. *For $N \ge \alpha + 1$ the integral* (3.20) *is norm convergent and $\forall h \in (0,1]$*

$$\|\Psi(-h^2\Delta_D)\|_{L^p(\Omega)\to L^p(\Omega)} \le c\|\Psi\|_{N+1}, \tag{3.23}$$

for some constant c independent of h.

Remark 3.13. Notice how the Mikhlin multiplier condition (2.2) on Ψ does not imply boundedness of $\|\Psi\|_{N+1}$: we need extra decay at infinity.

Proof: By scale invariance it is enough to prove (3.23) for $h = 1$. The integrand in (3.20) is norm continuous for $z \notin \mathbb{R}$. If we set

$$U \stackrel{\text{def}}{=} \{z = x + iy | \langle x \rangle < |y| < 2\langle x \rangle\}, \quad V \stackrel{\text{def}}{=} \{z = x + iy | 0 < |y| < 2\langle x \rangle\},$$

then the norm of the integrand is dominated by

$$c\sum_{m=0}^{N} |\partial^m \Psi(x)| \frac{2^m}{m!} \langle x \rangle^{m-2} \|\partial\tau\|_{L^\infty([1,2])} 1_U(x+iy) +$$
$$+ c|\partial^{N+1}\Psi(x)| \frac{2^N}{N!} |y|^N \left(\frac{\langle x \rangle}{|y|}\right)^\alpha \|\tau\|_{L^\infty([0,2])} 1_V(x+iy).$$

Integrating with respect to y for $N \ge \alpha + 1$ yields the bound

$$\|\Psi(-\Delta_D)\|_{L^p(\Omega)\to L^p(\Omega)} \lesssim \int_{\mathbb{R}} \Big(\sum_{m=0}^{N} |\partial^m \Psi(x)| \langle x \rangle^{m-1} +$$
$$+ |\partial^{N+1}\Psi(x)| \langle x \rangle^N \Big) dx = \|\Psi\|_{N+1}.$$

One may then prove that the operator $\Psi(-\Delta_D)$, acting on $L^p(\Omega)$, is independent of $N \ge 1 + n/2$ and of the cut-off function τ in the definition of $\tilde{\Psi}$, see [12].

We now recall two lemma which will be useful when composing operators.

Lemma 3.14 (Lemma 2.2.5,[12])**.** *If $\Psi \in C_0^\infty(\mathbb{R})$ has support disjoint from the spectrum of $-h^2\Delta_D$ then $\Psi(-h^2\Delta_D) = 0$.*

Lemma 3.15 (Lemma 2.2.6, [12]). *If Ψ_1, $\Psi_2 \in C_0^\infty(\mathbb{R})$, then $(\Psi_1\Psi_2)(-h^2\Delta_D) = \Psi_1(-h^2\Delta_D)\Psi_2(-h^2\Delta_D)$.*

For the remaining part of the Appendix we prove the resolvent estimate (3.22) from Lemma 3.11. If $\mathrm{Re} z > 0$, this is nothing but a standard elliptic estimate. The trouble comes with $\mathrm{Re} z < 0$ and getting close to the spectrum. In $\mathbb{R}^n$, one may evaluate directly the convolution operator by proving its kernel to be in L^1: this follows from

$$|z + |\xi|^2|^2 = \sin^2 \frac{(\pi - \theta)}{2}(|z| + |\xi|^2)^2 + \cos^2 \frac{(\pi - \theta)}{2}(|\xi|^2 - |z|)^2, \quad \text{with} \quad z = |z|e^{i\theta},$$

and a direct computation of L^2 norms of $\partial^\alpha(z + |\xi|^2)^{-1}$. By reflection, one then extends this estimate to the half-space case, with both Dirichlet and Neumann boundary conditions. By localizing L^p estimates close to the boundary and flattening, one may then obtain the desired estimate (3.22); such an approach is carried out in [1] in a greater generality (systems of Laplace equations, mixed boundary conditions), at the expense of fixing the angle θ and not tracking explicit dependances on $|z|$ and θ. While (relatively) elementary, such a proof is, out of necessity, filled with lenghty calculations and most certainly does not provide the sharpest constant. It is worth noting, however, that it relies on standard elliptic techniques.

To keep in line with the parabolic approach, we present a short proof, relying on the holomorphic nature of $S(w)$ in the half-plane $\mathrm{Re} w > 0$. Remark that by our L^p bound on $S(t)$, $t \in \mathbb{R}_+$, the trivial L^2 bound on $S(w)$, $\mathrm{Re} w \geq 0$, and Stein's parameter version of complex interpolation, one may easily derive that $S(w)$ is holomorphic in a sector around the positive real axis; but its angle will narrow with large or small p. However the argument may be refined and $S(w)$ was proved to be holomorphic in the whole right half-plane in [19], using in a crucial way the Gaussian nature of the heat kernel on domains ([10]). This was extented to more general settings in [8], where an explicit bound is stated:

$$\|S(w)\|_{L^p \to L^p} \leq C_\varepsilon \left(\frac{|w|}{|\mathrm{Re} w|}\right)^{n\left|\frac{1}{2} - \frac{1}{p}\right| + \varepsilon}. \tag{3.24}$$

Then (3.22) is a direct consequence of the following standard computation: recall the following formula, which is simply a Laplace transform,

$$(z - \Delta_D)^{-1} = \int_L e^{w\Delta_D - wz} dw, \tag{3.25}$$

where L can be chosen to be a half ray from the origin. Set $z = re^{i\theta}$, $w = \rho e^{i\phi}$, then

$$(z - \Delta_D)^{-1} = \int_0^{+\infty} e^{\rho \exp(i\phi)\Delta_D - r\rho \exp i(\theta + \phi)} d\rho.$$

Now, if $\mathrm{Re} z > 0$, we may take $\phi = 0$ and use estimates for the semi-group $S(\rho)$. We would like to extend the range to the $\mathrm{Re} z < 0$ region, up to a thin sector around the negative real axis ($|\pi - \theta| < \epsilon$); getting close to the spectrum is required if we want to define $\Psi(-\Delta_D)$ with $\Psi \in C_0^\infty(]0, +\infty[)$. One picks ϕ such that $2|\theta + \phi| < \pi$, which ensures a decaying exponential in 3.25, provided we bound $S(w)$ in L^p. But the condition on ϕ yields $|\phi| < \pi/2$, and the bound amounts to the holomorphy of $S(w)$. The constant in (3.24) translates into a $(|z|/|\mathrm{Im} z|)^\alpha$ factor, while integration over ρ provides the remaining $1/|\mathrm{Im} z|$ in (3.22). This concludes the proof.

References

[1] T. Akiyama, H. Kasai, Y. Shibata, and M. Tsutsumi. On a resolvent estimate of a system of Laplace operators with perfect wall condition. *Funkcial. Ekvac.*, 47(3):361–394, 2004.

[2] Georgios K. Alexopoulos. L^p bounds for spectral multipliers from Gaussian estimates of the heat kernel. unpublished manuscript, 1999.

[3] Georgios K. Alexopoulos. Spectral multipliers for Markov chains. *J. Math. Soc. Japan*, 56(3):833–852, 2004.

[4] Pascal Auscher. On necessary and sufficient conditions for L^p-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates. *Mem. Amer. Math. Soc.*, 186(871):xviii+75, 2007.

[5] Pascal Auscher, Alan McIntosh, and Philippe Tchamitchian. Heat kernels of second order complex elliptic operators and applications. *J. Funct. Anal.*, 152(1):22–73, 1998.

[6] Pascal Auscher and Philippe Tchamitchian. Square root problem for divergence operators and related topics. *Astérisque*, (249):viii+172, 1998.

[7] N. Burq, P. Gérard, and N. Tzvetkov. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. *Amer. J. Math.*, 126(3):569–605, 2004.

[8] Gilles Carron, Thierry Coulhon, and El-Maati Ouhabaz. Gaussian estimates and L^p-boundedness of Riesz means. *J. Evol. Equ.*, 2(3):299–317, 2002.

[9] Daniel Daners. Heat kernel estimates for operators with boundary conditions. *Math. Nachr.*, 217:13–41, 2000.

[10] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[11] E. B. Davies. Pointwise bounds on the space and time derivatives of heat kernels. *J. Operator Theory*, 21(2):367–378, 1989.

[12] E. B. Davies. The functional calculus. *J. London Math. Soc. (2)*, 52(1):166–176, 1995.

[13] E. B. Davies. Non-Gaussian aspects of heat kernel behaviour. *J. London Math. Soc. (2)*, 55(1):105–125, 1997.

[14] E. M. Dyn′kin. An operator calculus based on the Cauchy-Green formula. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 30:33–39, 1972. Investigations on linear operators and the theory of functions, III.

[15] B. Helffer and J. Sjöstrand. Équation de Schrödinger avec champ magnétique et équation de Harper. In *Schrödinger operators (Sønderborg, 1988)*, volume 345 of *Lecture Notes in Phys.*, pages 118–197. Springer, Berlin, 1989.

[16] Oana Ivanovici. On Schrodinger equation outside strictly convex obstacles, 2008. `arXiv:math.AP/0809.1060`.

[17] Gilles Lebeau. Estimation de dispersion pour les ondes dans un convexe. In *Journées "Équations aux Dérivées Partielles" (Evian, 2006)*. 2006.

[18] Francis Nier. A variational formulation of Schrödinger-Poisson systems in dimension $d \leq 3$. *Comm. Partial Differential Equations*, 18(7-8):1125–1147, 1993.

[19] El-Maati Ouhabaz. Gaussian estimates and holomorphy of semigroups. *Proc. Amer. Math. Soc.*, 123(5):1465–1474, 1995.

[20] Fabrice Planchon and Luis Vega. Bilinear identities and applications, 2007. to appear in Ann. Sci. E.N.S., `arXiv:math.AP/0712.4076`.

[21] A. Seeger and C. D. Sogge. On the boundedness of functions of (pseudo-) differential operators on compact manifolds. *Duke Math. J.*, 59(3):709–736, 1989.

[22] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[23] Xuan Thinh Duong, El Maati Ouhabaz, and Adam Sikora. Plancherel-type estimates and sharp spectral multipliers. *J. Funct. Anal.*, 196(2):443–485, 2002.

[24] François Trèves. *Introduction to pseudodifferential and Fourier integral operators. Vol. 1*. Plenum Press, New York, 1980. Pseudodifferential operators, The University Series in Mathematics.

[25] H. Triebel. *Interpolation theory, function spaces, differential operators*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[26] Hans Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1983.

[27] Hans Triebel. *Theory of function spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.

E-mail address: oana.ivanovici@math.u-psud.fr

Université Paris Sud, Mathématiques, Bât 430, 91405 Orsay Cedex

E-mail address: fab@math.univ-paris13.fr

Université Paris 13, L.A.G.A.,UMR 7539, Institut Galilée, 99 avenue J.B,Clément, F-93430 Villetaneuse